

# An evaluation of the central value of the automorphic scattering determinant

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## Abstract

Let  $M$  be a finite volume, non-compact hyperbolic Riemann surface, possibly with elliptic fixed points, and let  $\phi(s)$  denote the automorphic scattering determinant. From the known functional equation  $\phi(s)\phi(1-s) = 1$  one concludes that  $\phi(1/2)^2 = 1$ . However, except for the relatively few instances when  $\phi(s)$  is explicitly computable, one does not know  $\phi(1/2)$ . In this article we address this problem and prove the following result. Let  $N$  and  $P$  denote the number of zeros and poles, respectively, of  $\phi(s)$  in  $(1/2, \infty)$ , counted with multiplicities. Let  $d(1)$  be the coefficient of the leading term from the Dirichlet series component of  $\phi(s)$ . Then  $\phi(1/2) = (-1)^{N+P} \cdot \text{sgn}(d(1))$ .

## 1 Introduction

Various problems and conjectures in number theory can be equated to questions in harmonic analysis. For instance, the Riemann hypothesis for the Riemann zeta function is equivalent to the determination of the zeros of the meromorphic continuation of the (classical) non-parabolic Eisenstein series  $E(s, z)$  associated to  $\text{PSL}(2, \mathbb{Z})$ . The Lindelöf hypothesis for the Riemann zeta function is implied by a sup-norm bound for  $E(1/2 + ir, z)$  when  $r \in \mathbb{R}$ . As a result, harmonic analysis associated to quotients of the hyperbolic upper half plane  $\mathbb{H}$  has received considerable interest and study.

Nonetheless, there are many basic questions which remain unsolved. The far-reaching Phillips-Sarnak philosophy [9] asserts that only in presence of arithmetic or geometric symmetry will there exist square-integrable eigenfunctions of the Laplacian. Though there is compelling results supporting this point of view, the main conjectures remain unanswered. Within this framework, there are several other questions associated to the harmonic analysis of finite volume quotients of  $\mathbb{H}$  which are seemingly approachable yet their solutions remain elusive. The purpose of this paper is to address one such question.

Let  $\Gamma$  be a Fuchsian group of the first kind with  $c$  cusps, and assume  $c > 0$ . Let  $M = \Gamma \backslash \mathbb{H}$  be the finite volume, non-compact orbifold quotient space. Let  $\phi(s)$  denote the automorphic scattering determinant, meaning the determinant of the hyperbolic scattering matrix  $\Phi(s)$  which is obtained by computing the various Fourier expansions of the non-holomorphic, parabolic Eisenstein series associated to  $M$ . The function  $\phi(s)$  is meromorphic of order at most two. Furthermore,  $\phi(s)$  is holomorphic for  $\text{Re}(s) > \frac{1}{2}$ , except for a finite number of poles, and it satisfies the functional equation

$$\phi(s)\phi(1-s) = 1. \quad (1.1)$$

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The functional equation implies that  $\phi(1/2)^2 = 1$ . However, except for certain arithmetic groups, one does not know if  $\phi(1/2)$  is one or minus one. The main result of this article is to evaluate  $\phi(1/2)$  in terms of more elementary spectral and group theoretic data associated to  $M$ .

Let us recall the notation needed to state the main theorem. For  $\text{Re}(s) > 1$  the automorphic scattering determinant  $\phi(s)$  can be written as an absolutely convergent generalized Dirichlet series and Gamma functions. Specifically, for  $\text{Re}(s) > 1$  we have that

$$\phi(s) = \pi^{\frac{s}{2}} \left( \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right)^c \sum_{n=1}^{\infty} \frac{d(n)}{g_n^{2s}} \quad (1.2)$$

where  $0 < g_1 < g_2 < \dots$  and  $d(n) \in \mathbb{R}$  with  $d(1) \neq 0$ .

**Theorem 1.1.** *Let  $N$  and  $P$  denote the number of zeros and poles, respectively, of  $\phi(s)$  in  $[1/2, \infty)$ , counted with multiplicities. Let  $d(1)$  be as in (1.2). Then*

$$\phi(1/2) = (-1)^{N+P} \text{sgn}(d(1)),$$

where  $\text{sgn}$  denotes the sign of a real number.

We give examples of arithmetic groups where  $\phi(1/2) = 1$  and where  $\phi(1/2) = -1$ . However, it remains to be seen if  $\phi(1/2)$  is constant on any given moduli space, and, if not, then how the values of  $\phi(1/2)$  partition the moduli space. We leave this problem to the interested reader.

## 2 Preliminary material

### 2.1 Basic notation

Let  $\Gamma \subseteq \text{PSL}(2, \mathbb{R})$  be a Fuchsian group of the first kind acting by fractional linear transformations on the upper half-plane  $\mathbb{H} := \{z \in \mathbb{C} \mid z = x + iy, y > 0\}$ . Let  $M$  be the quotient space  $\Gamma \backslash \mathbb{H}$  and  $g$  the genus of  $M$ . Denote by  $\mathbf{c}$  number of inequivalent cusps of  $M$  and by  $\{R\}_{\Gamma}$  the set of inequivalent elliptic classes of elements of  $\Gamma$ . For a fixed elliptic representative  $R$ , we denote by  $d_R$  the order of element  $R$  and by  $\mathbf{e}$  the cardinality of the finite set  $\{R\}_{\Gamma}$  of inequivalent elliptic classes in  $\Gamma$ .

Recall that the hyperbolic volume  $\text{vol}(M)$  of  $M$  is given by the Gauss-Bonnet formula

$$\text{vol}(M) = 2\pi \left( 2g - 2 + \mathbf{c} + \sum_{\{R\}_{\Gamma}} \left( 1 - \frac{1}{d_R} \right) \right).$$

Given a meromorphic function  $f(s)$ , we define the *null set*  $N(f)$  to be  $N(f) = \{s \in \mathbb{C} \mid f(s) = 0\}$  counted with multiplicity. Similarly,  $P(f)$  denotes the *polar set*, the set of points where  $f$  has a pole.

### 2.2 The Gamma function

Let  $\Gamma(s)$  denote the Gamma function. Its poles are all simple and located at each point of  $-\mathbb{N}$ , where  $-\mathbb{N} = \{0, -1, -2, \dots\}$ . For  $|\arg s| \leq \pi - \delta$  and  $\delta > 0$ , the asymptotic expansion [2, p. 20] of  $\log \Gamma(s)$  is given by

$$\log \Gamma(s) = \frac{1}{2} \log 2\pi + \left( s - \frac{1}{2} \right) \log s - s + \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j-1)2j} \frac{1}{s^{2j-1}} + g_m(s). \quad (2.1)$$

Here  $B_i$  are the Bernoulli numbers. Also, for each  $m$ ,  $g_m(s)$  is a holomorphic function in the right half plane  $\text{Re}(s) > 0$  such that  $g_m^{(j)}(s) = O(s^{-2m+1-j})$  as  $\text{Re}(s) \rightarrow \infty$  for all integers  $j \geq 0$ , and where the implied constant depends on  $j$  and  $m$ .

### 2.3 The double Gamma function

The Barnes double Gamma function is an entire order two function defined by

$$G(s+1) = (2\pi)^{s/2} \exp \left[ -\frac{1}{2} [(1+\gamma)s^2 + s] \right] \prod_{n=1}^{\infty} \left( 1 + \frac{s}{n} \right)^n \exp \left[ -s + \frac{s^2}{2n} \right],$$

where  $\gamma$  is the Euler constant. Therefore,  $G(s+1)$  has a zero of multiplicity  $n$ , at each point  $-n \in \{-1, -2, \dots\}$ . For  $\operatorname{Re}(s) > 0$  and as  $s \rightarrow \infty$ , the asymptotic expansion of  $\log G(s+1)$  is given in [3] or [1, Lemma 5.1] by

$$\log G(s+1) = \frac{s^2}{2} \left( \log s - \frac{3}{2} \right) - \frac{\log s}{12} - s \zeta'(0) + \zeta'(-1) - \sum_{k=1}^n \frac{B_{2k+2}}{4k(k+1)s^{2k}} + h_{n+1}(s). \quad (2.2)$$

Here,  $\zeta(s)$  is the Riemann zeta-function and

$$h_{n+1}(s) = \frac{(-1)^{n+1}}{s^{2n+2}} \int_0^{\infty} \frac{t}{\exp(2\pi t) - 1} \int_0^{t^2} \frac{y^{n+1}}{y + s^2} dy dt.$$

By a close inspection of the proof of [1, Lemma 5.1], it follows that  $h_{n+1}(s)$  is holomorphic function in the right half plane  $\operatorname{Re}(s) > 0$  which satisfies the asymptotic relation  $h_{n+1}^{(j)}(s) = O(s^{-2n-2-j})$  as  $\operatorname{Re}(s) \rightarrow \infty$  for all integers  $j \geq 0$ , and where the implied constant depends upon  $j$  and  $n$ .

Using the Gamma and Barnes double Gamma function, we can construct a function whose divisor coincides with the trivial zeros and poles of the Selberg zeta function (see Equation (3.5)), which we will define later. Specifically, we are interested in defining a meromorphic function whose poles are at the negative integers  $-n \in -\mathbb{N}$  with multiplicity

$$m_n = (2n+1)(2g-2+\mathbf{c}) + 2n\mathbf{e} - 2 \sum_{\{R\}_{\Gamma}} \lfloor \frac{n}{d_R} \rfloor,$$

where  $\lfloor x \rfloor$  denotes the integer part of a real number  $x$ . To do so, first observe that the function

$$\prod_{m=0}^{d_R-1} G\left(\frac{s+m}{d_R} + 1\right)$$

has zeros at  $-n \in -\mathbb{N}$  of order  $\lfloor \frac{n}{d_R} \rfloor$ . Hence, we define

$$G_E(s) = \prod_{\{R\}_{\Gamma}} \prod_{m=0}^{d_R-1} G\left(\frac{s+m}{d_R} + 1\right),$$

and set

$$G_1(s) = \left( \frac{(2\pi)^{-s}(G(s+1))^2}{\Gamma(s)} \right)^{2g-2+\mathbf{c}} \cdot ((2\pi)^{-s}(G(s+1))^2)^{\mathbf{e}} \cdot (G_E(s))^{-2}. \quad (2.3)$$

It is elementary to show that  $G_1(s)$  is an entire function of order two with zeros at points  $-n \in -\mathbb{N}$  and corresponding multiplicities  $m_n$ .

## 3 Zeta functions

We are further establishing notation by citing material from the well-known sources [6], [7] and [10].

### 3.1 Automorphic scattering determinant

We will rewrite (1.2) in a slightly different form. Let  $c_1 = -2 \log g_1 \neq 0$ ,  $c_2 = \log d(1)$ , and let  $u_n = g_n/g_1 > 1$ . Then for  $\operatorname{Re}(s) > 1$  we can write  $\phi(s) = L(s)H(s)$  where

$$L(s) = \pi^{\frac{s}{2}} \left( \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \right)^c e^{c_1 s + c_2} \quad (3.1)$$

and

$$H(s) = 1 + \sum_{n=2}^{\infty} \frac{a(n)}{u_n^{2s}}, \quad (3.2)$$

where  $a(n) \in \mathbb{R}$ . The series (3.2) converges absolutely for  $\operatorname{Re}(s) > 1$ . From the generalized Dirichlet series representation (3.2) of  $H(s)$ , it follows that

$$\frac{d^k}{ds^k} \log H(s) = O(\beta_k^{-\operatorname{Re}(s)}) \quad \text{when} \quad \operatorname{Re}(s) \rightarrow +\infty, \quad (3.3)$$

for some  $\beta_k > 1$  where the implied constant depends on  $k \in \mathbb{N}$ .

The divisor of  $\phi(s)$  consists of the following sets of points:

1. Finitely many real zeros of the form  $1 - \sigma_i \in [0, 1/2)$  for  $i = 1 \dots T$ , each with multiplicity  $q(\sigma_i)$ ;
2. Finitely many real zeros of the form  $\rho_i > 1/2$ ,  $i = 1 \dots N$ , where  $N$  is defined to be the sum of the multiplicities;
3. Finitely many real poles of the form  $1 - \rho_i < 1/2$ , where  $i = 1 \dots N$ ;
4. Finitely many poles  $\sigma_i \in (1/2, 1]$ , where by the functional equation (1.1) each pole has multiplicity  $q(\sigma_i)$  and whose sum is  $P$ ;
5. Poles of the form  $1 - \rho$  and  $1 - \bar{\rho}$  with  $\operatorname{Re}(\rho) > 1/2$  and  $\operatorname{Im}(\rho) > 0$ ;
6. Zeros of the form  $\rho$  and  $\bar{\rho}$  with  $\operatorname{Re}(\rho) > 1/2$  and  $\operatorname{Im}(\rho) > 0$ .

Let  $\lambda_i$  be an eigenvalue for the positive, self-adjoint extension  $\Delta$  of the hyperbolic Laplacian. Denote by  $A(\lambda_i)$  the  $\Delta$ -eigenspace corresponding to the eigenvalue  $\lambda_i$ . Set  $A_1(\lambda_i)$  to be the subspace of  $A(\lambda_i)$  that is spanned by the incomplete theta series. For each pole  $\sigma_i \in (1/2, 1]$ ,  $i = 1, \dots, T$  the space  $A_1(\sigma_i(1 - \sigma_i))$  is non-trivial. In fact from ([6, Eq. 3.33 on p.299]) we have that

$$q(\sigma_i) = [\text{The multiplicity of the pole of } \phi(s) \text{ at } s = \sigma_i] \leq \dim A_1(\sigma_i(1 - \sigma_i)) \leq c.$$

The eigenvalue  $\lambda_i = \sigma_i(1 - \sigma_i)$  is called a *residual eigenvalue*.

### 3.2 Selberg zeta-function

The Selberg zeta function associated to the quotient space  $M = \Gamma \backslash \mathbb{H}$  is defined for  $\operatorname{Re}(s) > 1$  by the absolutely convergent Euler product

$$Z(s) = \prod_{\{P_0\} \in P(\Gamma)} \prod_{n=0}^{\infty} \left( 1 - N(P_0)^{-(s+n)} \right),$$

where  $P(\Gamma)$  denotes the set of all primitive hyperbolic conjugacy classes in  $\Gamma$ , and  $N(P_0)$  denotes the norm of  $P_0 \in \Gamma$ . From the product representation given above, we have for  $\operatorname{Re}(s) > 1$  that

$$\log Z(s) = \sum_{\{P_0\} \in P(\Gamma)} \sum_{n=0}^{\infty} \left( - \sum_{l=1}^{\infty} \frac{N(P_0)^{-(s+n)l}}{l} \right) = - \sum_{P \in H(\Gamma)} \frac{\Lambda(P)}{N(P)^s \log N(P)},$$

where  $H(\Gamma)$  denotes the set of all hyperbolic conjugacy classes in  $\Gamma$ , and  $\Lambda(P) = \frac{\log N(P_0)}{1-N(P)^{-1}}$ , for the primitive element  $P_0$  in the conjugacy class containing  $P$ .

Let  $P_{00}$  be the primitive hyperbolic conjugacy class in all of  $P(\Gamma)$  with the smallest norm. Setting  $\alpha = N(P_{00})^{\frac{1}{2}}$ , for  $\operatorname{Re}(s) > 2$  and  $k \in \mathbb{N}$  we have the following asymptotic formula

$$\frac{d^k}{ds^k} \log Z(s) = O(\alpha^{-\operatorname{Re}(s)}) \quad \text{when} \quad \operatorname{Re}(s) \rightarrow +\infty, \quad (3.4)$$

with an implied constant which depends on  $k \in \mathbb{N}$ .

If  $\lambda_j$  is an eigenvalue in the discrete spectrum of  $\Delta$ , let  $m(\lambda_j)$  denote its multiplicity. We now state the divisor of the  $Z(s)$  (see [11, p. 49] [6, p. 499]):

1. Zeros at the points  $s_j$  on the line  $\operatorname{Re}(s) = \frac{1}{2}$  symmetric relative to the real axis and in  $(1/2, 1]$ , where each zero  $s_j$  has multiplicity  $m(s_j) = m(\lambda_j)$  where  $s_j(1 - s_j) = \lambda_j$  is an eigenvalue in the discrete spectrum of  $\Delta$ ;
2. Zeros at the points  $s_j \in [0, 1/2)$  where  $s_j(1 - s_j) = \lambda_j \in [0, 1/4)$  is an eigenvalue in the discrete spectrum of  $\Delta$  and the multiplicity  $\tilde{m}(s_j)$  is given by  $\tilde{m}(s_j) = m(\lambda_j) - q(1 - s_j) \geq 0$ ; we denote by  $K$  the number of eigenvalues  $\lambda_j \in [0, 1/4)$  and put  $m_j = m(\lambda_j)$ ,  $j = 1, \dots, K$ .  
Note that, in the case when  $\lambda_j$  is not the residual eigenvalue, we take  $q(1 - s_j) = 0$ , i.e.  $\tilde{m}(s_j) = m(\lambda_j)$ .
3. The point  $s = \frac{1}{2}$  can be a zero or a pole, and the order of the point as a divisor is

$$\mathfrak{a} = 2d_{1/4} - \frac{1}{2} (\mathfrak{c} - \operatorname{tr} \Phi(\frac{1}{2}))$$

where  $d_{1/4}$  be the multiplicity of the possible eigenvalue  $\lambda = \frac{1}{4}$  of  $\Delta$ ;

4. Poles at  $s = -n - \frac{1}{2}$ , where  $n = 0, 1, 2, \dots$ , each with multiplicity  $\mathfrak{c}$ ;
5. Finitely many real zeros  $1 - \rho_i < 1/2$ , where  $i = 1 \dots N$ ;
6. Zeros at each  $s = 1 - \rho, 1 - \bar{\rho}$  where  $\rho$  is a zero of  $\phi(s)$  with  $\operatorname{Re}(\rho) > \frac{1}{2}$  and  $\operatorname{Im}(\rho) > 0$ ;
7. Zeros at points  $s = -n \in -\mathbb{N}$ , with multiplicities

$$m_n = \frac{\operatorname{vol}(M)}{2\pi} (2n + 1) - \sum_{\{R\}_\Gamma} \frac{1}{d_R} \sum_{k=1}^{d_R-1} \frac{\sin\left(\frac{k\pi(2n+1)}{d_R}\right)}{\sin\left(\frac{k\pi}{d_R}\right)}.$$

The last set of zeros are called *trivial* zeros. It is possible to write the multiplicity  $m_n$  in a different way. By using a double induction in the variables  $n$  and  $d_R$ , one can show that

$$\sum_{\{R\}_\Gamma} \frac{1}{d_R} \left( 2n + 1 + \sum_{k=1}^{d_R-1} \frac{\sin\left(\frac{k\pi(2n+1)}{d_R}\right)}{\sin\left(\frac{k\pi}{d_R}\right)} \right) = \sum_{\{R\}_\Gamma} \left( 2 \left\lfloor \frac{n}{d_R} \right\rfloor + 1 \right).$$

Therefore, by applying the Gauss-Bonnet formula for  $\operatorname{vol}(M)$  we immediately get

$$m_n = (2n + 1)(2g - 2 + \mathfrak{c}) + 2n\mathfrak{e} - 2 \sum_{\{R\}_\Gamma} \left\lfloor \frac{n}{d_R} \right\rfloor. \quad (3.5)$$

### 3.3 Complete zeta functions

Define

$$Z_+(s) = \frac{Z(s)}{G_1(s)(\Gamma(s-1/2))^c},$$

where  $G_1(s)$  is defined by (2.3). Note that we have canceled out the trivial zeros and poles of  $Z(s)$ . Hence the zero set  $N(Z_+)$  of  $Z_+$  consists of the following points:

1. At  $s = \frac{1}{2}$  with multiplicity  $\mathfrak{a}$  where

$$\mathfrak{a} = 2d_{1/4} + \mathfrak{c} - \frac{1}{2} (\mathfrak{c} - \text{tr } \Phi(\frac{1}{2})) = 2d_{1/4} + \frac{1}{2} (\mathfrak{c} + \text{tr } \Phi(\frac{1}{2})) \geq 0;$$

2. At the points  $s_j \in [0, 1/2)$  where  $s_j(1-s_j) = \lambda_j$  is an eigenvalue in the discrete spectrum of  $\Delta$  each with multiplicity  $m(\lambda_j) - q(1-s_j) \geq 0$ ;
3. At the points  $s_j$  on the line  $\text{Re}(s) = \frac{1}{2}$  symmetric relative to the real axis and in  $(1/2, 1]$  where each zero  $s_j$  has multiplicity  $m(s_j) = m(\lambda_j)$  where  $s_j(1-s_j) = \lambda_j$  is an eigenvalue in the discrete spectrum of  $\Delta$ ;
4. At each point  $s = 1 - \rho, 1 - \bar{\rho}$  where  $\rho$  is a zero of  $\phi(s)$  with  $\text{Re}(\rho) > \frac{1}{2}$ , and  $\text{Im}(\rho) > 0$ .

Sets one, two and three in the above enumeration are finite and are such that all zeros are real. Within set four, there are a finite number of real zeros. Hence, in total there are a finite number of real zeros of  $Z_+$ , and the location of the zeros are described in the above four sets.

Define  $Z_-(s) = Z_+(s)\phi(s)$ . It follows that  $N(Z_-) = 1 - N(Z_+)$ . In other words,  $s$  is a zero of  $Z_+$  if and only if  $1-s$  is a zero, necessarily with the same multiplicity, of  $Z_-$ .

## 4 Superzeta functions

### 4.1 Regularized products using superzeta functions

Let  $\mathbb{R}^- = (-\infty, 0]$  be the non-positive real numbers. Let  $\{y_k\}_{k \in \mathbb{N}}$  be the sequence of zeros of an entire function  $f$  of order at most two, repeated with their multiplicities. Let

$$X_f = \{z \in \mathbb{C} \mid (z - y_k) \notin \mathbb{R}^- \text{ for all } y_k\}.$$

For  $z \in X_f$ , and  $s \in \mathbb{C}$  consider the series

$$\mathcal{Z}_f(s, z) = \sum_{k=1}^{\infty} (z - y_k)^{-s}, \quad (4.1)$$

where the complex exponent is defined using the principal branch of the logarithm with  $\arg z \in (-\pi, \pi)$  in the cut plane  $\mathbb{C} \setminus \mathbb{R}^-$ . Since  $f$  is of order at most two, the series  $\mathcal{Z}_f(s, z)$  converges absolutely for  $\text{Re}(s) > 2$ . Following [12], the series  $\mathcal{Z}_f(s, z)$  is called the *superzeta function* associated to the zeros of  $f$ , or the simply the *superzeta function* of  $f$ .

If  $\mathcal{Z}_f(s, z)$  has a meromorphic continuation which is regular at  $s = 0$ , we define the *superzeta regularized product* associated to  $f$  as

$$D_f(z) = \exp \left( -\frac{d}{ds} \mathcal{Z}_f(s, z) \Big|_{s=0} \right).$$

Hadamard's product formula allows us to write

$$f(z) = \Delta_f(z) = e^{g(z)} z^r \prod_{k=1}^{\infty} \left( \left( 1 - \frac{z}{y_k} \right) \exp \left[ \frac{z}{y_k} + \frac{z^2}{2y_k^2} \right] \right), \quad (4.2)$$

where  $g(z)$  is a polynomial of degree 2 or less,  $r \geq 0$  is the order of eventual zero of  $f$  at  $z = 0$ , and the other zeros  $y_k$  are listed with multiplicity.

The following proposition, originally due to Voros ([12], [14], [15]) is proven in [5, Prop. 4.1]:

**Proposition 4.1.** *Let  $f$  be an entire function of order two, and for  $k \in \mathbb{N}$ , let  $y_k$  be the sequence of zeros of  $f$ . Let  $\Delta_f(z)$  denote the Hadamard product representation of  $f$ . Assume that for  $n > 2$  we have the following asymptotic expansion:*

$$\log \Delta_f(z) = \tilde{a}_2 z^2 (\log z - \frac{3}{2}) + b_2 z^2 + \tilde{a}_1 z (\log z - 1) + b_1 z + \tilde{a}_0 \log z + b_0 + \sum_{k=1}^{n-1} a_k z^{\mu_k} + h_n(z), \quad (4.3)$$

where  $1 > \mu_1 > \dots > \mu_n \rightarrow -\infty$ , and  $h_n(z)$  is a sequence of holomorphic functions in the sector  $|\arg z| < \theta < \pi$ , ( $\theta > 0$ ) such that  $h_n^{(j)}(z) = O(|z|^{\mu_n - j})$ , as  $|z| \rightarrow \infty$  in the above sector, for all integers  $j \geq 0$ .

Then, for all  $z \in X_f$ , the superzeta function  $\mathcal{Z}_f(s, z)$  has a meromorphic continuation to the half-plane  $\operatorname{Re}(s) < 2$  which is regular at  $s = 0$ .

Furthermore, the superzeta regularized product  $D_f(z)$  associated to  $f(s)$  is related to  $\Delta_f(z)$  through the formula

$$\exp \left( -\frac{d}{ds} \mathcal{Z}_f(s, z) \Big|_{s=0} \right) = D_f(z) = e^{-(b_2 z^2 + b_1 z + b_0)} \Delta_f(z). \quad (4.4)$$

## 4.2 Superzeta functions associated to $Z_+$ and $Z_-$

Let  $X_{\pm} = X_{Z_{\pm}}$ , and for  $z \in X_{\pm}$ , denote by  $\mathcal{Z}_{\pm}(s, z) := \mathcal{Z}_{Z_{\pm}}(s, z)$  the superzeta functions of  $Z_{\pm}$ .

**Theorem 4.2.** *For  $z \in X_{\pm}$ , the superzeta functions  $\mathcal{Z}_{\pm}(s, z)$  have meromorphic continuations to all of  $s \in \mathbb{C}$ , regular at  $s = 0$ . Furthermore, for  $z \in X_+ \cap X_-$*

$$\phi(z) = (\pi)^{\frac{c}{2}} e^{c_1 z + c_2} \exp \left( -\frac{d}{ds} (\mathcal{Z}_-(s, z) - \mathcal{Z}_+(s, z)) \Big|_{s=0} \right). \quad (4.5)$$

*Proof.* We claim that  $Z_+(s)$  and  $Z_-(s)$  both are entire, order two functions which satisfy the hypothesis of Proposition 4.1. Indeed, the function  $G_1(s)$  is a product of rescaled Barnes double Gamma functions, so by using the asymptotic expansion (2.1) of the Gamma function, the expansion (2.2) of the Barnes double Gamma function, the bound (3.4) for logarithm of the Selberg zeta function, and the asymptotic expansion of the logarithm of the automorphic scattering matrix  $\phi(s) = L(s)H(s)$ , deduced from (3.1) and (3.3), we can obtain an asymptotic expansion of the form (4.3) for both  $Z_+$  and  $Z_-$ . We refer to the proof of [5, Thm. 6.2] where similar computations are worked out in complete detail. Thus, by Proposition 4.1 both  $\mathcal{Z}_{\pm}(s, z)$  have meromorphic continuations to all of  $s \in \mathbb{C}$  which are regular at  $s = 0$ .

Recall that  $Z_-(s) = \phi(s)Z_+(s)$ . Hence from the asymptotic properties of  $\log \phi(s)$ , it follows that

$$\log Z_-(s) = \log Z_+(s) + \frac{c}{2} \log \pi - \frac{c}{2} \log s + c_1 s + c_2 + o(1) \quad \text{as } s \rightarrow \infty.$$

Here  $c_1$  and  $c_2$  are from (3.1), and we used the asymptotic expansion of  $\log \Gamma(s - \frac{1}{2})$  which can be obtained from (2.1) and Legendre's duplication formula. Since  $\log Z_+(s)$  has an expansion of the form

$$\log Z_+(s) = \tilde{a}_2 s^2 (\log s - \frac{3}{2}) + b_2 s^2 + \tilde{a}_1 s (\log s - 1) + b_1 s + \tilde{a}_0 \log s + b_0 + \sum_{k=1}^{n-1} a_k s^{\mu_k} + h_n(s),$$

we conclude that

$$\log Z_-(s) = \tilde{a}_2 s^2 (\log s - \frac{3}{2}) + b_2 s^2 + \tilde{a}_1 s (\log s - 1) + (b_1 + c_1) s + \tilde{a}'_0 \log s + (b_0 + \frac{c}{2} \log \pi + c_2) + \sum_{k=1}^{n-1} a'_k s^{\mu_k} + g_n(z).$$

Note that of the leading terms only  $\tilde{a}_0$ ,  $b_1$ , and  $b_0$  changed; however, only the  $b$ -terms are explicitly present on the right hand side of (4.4). Hence applying (4.4) to both  $Z_-(s)$ ,  $Z_+(s)$ , and recalling that  $Z_-(s) = \phi(s)Z_+(s)$  gives us (4.5).  $\square$

## 5 Proof of the main Theorem

As above, let  $K$  denote the number of exceptional eigenvalues  $\lambda_j$  of the Laplacian, let  $s_j \in (1/2, 1]$  be such that  $s_j(1-s_j) = \lambda_j$  and let  $m_j = m(\lambda_j)$  be the multiplicity of the eigenvalue  $\lambda_j$ ,  $j = 1, \dots, K$ .

For any  $j \in \{1, \dots, K\}$ ,  $s_j$  is a zero of  $Z_+(s)$  of multiplicity  $m_j$  and  $(1-s_j) \in [0, 1/2)$  is a zero of  $Z_+(s)$  of multiplicity  $m_j - q(s_j)$  (where we put  $q(s_j) = 0$  in case when  $s_j$  is not a pole of  $\phi(s)$ ). Moreover, recall that  $\rho_i > 1/2$ ,  $i = 1, \dots, N$  are real zeros of  $\phi(s)$ , counted according to their multiplicities and that the number of poles  $\sigma_i \in (1/2, 1]$  of  $\phi(s)$  is  $T \leq K$ .

We define

$$\mathcal{Z}_+^*(s, z) := \mathcal{Z}_+(s, z) - \frac{\mathfrak{a}}{(z-1/2)^s} - \sum_{j=1}^K \frac{m_j}{(z-s_j)^s} - \sum_{j=1}^K \frac{m_j - q(s_j)}{(z-(1-s_j))^s} - \sum_{i=1}^N \frac{1}{(z-(1-\rho_i))^s}$$

and

$$\mathcal{Z}_-^*(s, z) := \mathcal{Z}_-(s, z) - \frac{\mathfrak{a}}{(z-1/2)^s} - \sum_{j=1}^K \frac{m_j - q(s_j)}{(z-s_j)^s} - \sum_{j=1}^K \frac{m_j}{(z-(1-s_j))^s} - \sum_{i=1}^N \frac{1}{(z-\rho_i)^s}.$$

From Section 3.3 and proof of Theorem 4.2, we see that, for any fixed  $z \in (X_+ \cap X_-) \cup [1/2, 1]$  the functions  $\mathcal{Z}_\pm^*(s, z)$  are meromorphic in the entire  $s$ -plane, holomorphic at  $s = 0$ . Hence, by analytic continuation for  $z \in \mathbb{C}$ , equation (4.5) becomes

$$\begin{aligned} \phi(1/2) &= (\pi)^{\frac{\mathfrak{c}}{2}} e^{(c_1/2 + c_2)} \cdot \exp \left[ -\frac{d}{ds} \left( \mathcal{Z}_-^*(s, \tfrac{1}{2}) - \mathcal{Z}_+^*(s, \tfrac{1}{2}) \right) \right. \\ &\quad \left. + \sum_{i=1}^T q(\sigma_i) \left( \frac{1}{(\sigma_i - \frac{1}{2})^s} - \frac{1}{(\frac{1}{2} - \sigma_i)^s} \right) + \sum_{i=1}^N \left( \frac{1}{(\frac{1}{2} - \rho_i)^s} - \frac{1}{(\rho_i - \frac{1}{2})^s} \right) \right] \Big|_{s=0}. \end{aligned} \quad (5.1)$$

Note that the support of  $q(z)$  is on the set  $\{\sigma_1, \sigma_2, \dots, \sigma_T\}$ . Using the principal branch of  $\log(z)$ , where  $-\pi < \arg(z) \leq \pi$ , for  $0 \neq z \in \mathbb{R}$ , it is elementary that

$$\exp \left( \frac{d}{ds} z^{-s} \Big|_{s=0} \right) = \frac{1}{z}.$$

Recalling the definition of  $c_1, c_2$  from above, we can simplify (5.1) and write

$$\begin{aligned} \phi(1/2) &= (\pi)^{\frac{\mathfrak{c}}{2}} \frac{d(1)}{g_1} \exp \left( -\frac{d}{ds} \left( \mathcal{Z}_-^*(s, \tfrac{1}{2}) - \mathcal{Z}_+^*(s, \tfrac{1}{2}) \right) \Big|_{s=0} \right) \prod_{i=1}^T \left( \frac{\sigma_i - \frac{1}{2}}{\frac{1}{2} - \sigma_i} \right)^{q(\sigma_i)} \prod_{i=1}^N \left( \frac{\frac{1}{2} - \rho_i}{\rho_i - \frac{1}{2}} \right) \\ &= (-1)^{P+N} \frac{d(1)}{g_1} (\pi)^{\frac{\mathfrak{c}}{2}} \exp \left( -\frac{d}{ds} \left( \mathcal{Z}_-^*(s, \tfrac{1}{2}) - \mathcal{Z}_+^*(s, \tfrac{1}{2}) \right) \Big|_{s=0} \right) \end{aligned} \quad (5.2)$$

Next let us look at the difference  $\mathcal{Z}_-^*(s, 1/2) - \mathcal{Z}_+^*(s, 1/2)$ . For  $\operatorname{Re}(s) > 2$ ,

$$\mathcal{Z}_-^*(s, 1/2) = \sum_{\rho} \frac{1}{(\rho - 1/2)^s} + \sum_{j=1}^{\infty} \left( \frac{1}{(it_j)^s} + \frac{1}{(-it_j)^s} \right)$$

where the first sum is taken over all zeros  $\rho$  of the scattering determinant  $\phi(s)$  with  $\operatorname{Re}(\rho) > 1/2$  and  $\operatorname{Im}(\rho) \neq 0$  while the second sum is taken over all real  $t_j > 0$  such that  $\lambda_j = 1/4 + t_j^2 > 1/4$



are discrete eigenvalues of  $\Delta$ . Since the non-real zeros of  $Z_-(s)$  come in complex conjugate pairs, for  $\operatorname{Re}(s) > 2$  we may write

$$\sum_{\rho: \operatorname{Im}(\rho) > 0} \left( \frac{1}{(\rho - 1/2)^s} + \frac{1}{(\bar{\rho} - 1/2)^s} \right) + \sum_{j=1}^{\infty} \left( \frac{1}{(it_j)^s} + \frac{1}{(-it_j)^s} \right).$$

Elementary computations show that for real  $s > 2$ ,  $Z_-^*(s, 1/2)$  is real. By uniqueness of analytic continuation, we deduce that

$$\frac{d}{ds} Z_-^*(s, 1/2)|_{s=0}$$

is also real. Analogously, since the non-real zeros of  $Z_+(s)$  also come in complex-conjugate pairs, we deduce that

$$\frac{d}{ds} Z_+^*(s, 1/2)|_{s=0}$$

is also real. Hence

$$\exp \left( -\frac{d}{ds} (Z_-^*(s, 1/2) - Z_+^*(s, 1/2))|_{s=0} \right) = e^\alpha > 0,$$

for some  $\alpha \in \mathbb{R}$ . Substituting into (5.2), we get that

$$\phi(1/2) = (-1)^{N+P} \pi^{c/2} \frac{d(1)}{g_1} e^\alpha. \quad (5.3)$$

Since we know that  $\phi(1/2)^2 = 1$ , it remains to determine the sign of the above expression. However, since  $g_1 > 0$  and  $e^\alpha$  is positive, we conclude that  $\phi(1/2) = (-1)^{N+P} \cdot \operatorname{sgn}(d(1))$ , which completes the proof of the main theorem.

By taking the absolute values of both sides of (5.3), we obtain the following corollary.

**Corollary 5.1.** *With notation as above, we have that*

$$\exp \left( -\frac{d}{ds} (Z_-^*(s, 1/2) - Z_+^*(s, 1/2))|_{s=0} \right) = \frac{g_1}{\pi^{c/2} |d(1)|}.$$

## 6 Examples

**Example 1.** In the case when  $\Gamma$  is the modular group  $\operatorname{PSL}(2, \mathbb{Z})$ , the scattering determinant is given by

$$\phi(s) = \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \frac{\zeta(2s - 1)}{\zeta(2s)},$$

where  $\zeta(s)$  is the Riemann zeta function, hence

$$\phi(1/2) = \frac{\sqrt{\pi}}{\Gamma(1/2)} \cdot \zeta(0) \cdot \lim_{s \rightarrow 1/2} \frac{\Gamma(s + 1/2)}{(s - 1/2)\zeta(2s)} = -\frac{1}{2} \cdot \frac{1}{1/2} = -1.$$

This agrees with Theorem 1.1 since in this case,  $P = 1$  (there is only one residual eigenvalue of multiplicity one) and there are no real zeros of  $\phi(s)$  which are bigger than  $1/2$ , i.e.  $N = 0$ .

**Example 2.** Let  $N$  be a square-free number with  $r \geq 1$  distinct prime factors  $p_1, \dots, p_r$ . (Note: This  $N$  is not the same  $N$  as in the statement of the main theorem.) When  $\Gamma = \Gamma_0(N)$  is the congruence group, then the scattering matrix  $\phi_{N,0}(s)$  is given by

$$\phi_{N,0}(s) = \left[ \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \frac{\zeta(2s - 1)}{\zeta(2s)} \right]^{2^r} \prod_{p|N} \left( \frac{1 - p^{2-2s}}{1 - p^{2s}} \right)^{2^{r-1}},$$

see [6], formula (4.7) on p. 538, hence, obviously  $\phi(1/2) = 1$ .

On the other hand, the first factor in the above equation has a pole at  $s = 1$  of order  $2^r$ , while the second factor has a zero at  $s = 1$  of order  $r2^{r-1}$  and there are no other zeros or poles of  $\phi(s)$  that belong to the interval  $(1/2, +\infty)$ . Therefore, for a prime level  $N = p$ , meaning when  $r = 1$ , one has  $(-1)^{N+P} = -1$  indicating that one has  $\text{sgn}(d(1)) = -1$ . Indeed, if one expands the function

$$H(s) = \left[ \frac{\zeta(2s-1)}{\zeta(2s)} \right]^2 \left( \frac{1-p^{2-2s}}{1-p^{2s}} \right)$$

into Dirichlet series, it is easy to see that, actually,  $d(1) = -1$ . In case when  $N$  possesses two or more distinct prime factors, the number of real zeros and poles of the automorphic scattering determinant greater than  $1/2$  is even, and  $d(1)$  is evidently positive. Therefore, our main result is verified in this case.

**Example 3.** Let  $\{p_i\}$ , with  $i = 1, \dots, r$ , be a set of distinct primes and set  $N = p_1 \cdots p_r$ . The subset of  $\text{SL}(2, \mathbb{R})$ , defined by

$$\Gamma_0(N)^+ := \left\{ e^{-1/2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) : ad - bc = e, \ a, b, c, d, e \in \mathbb{Z}, \ e \mid N, \ e \mid a, \ e \mid d, \ N \mid c \right\}$$

is an arithmetic subgroup of  $\text{SL}(2, \mathbb{R})$ . In effect,  $\Gamma_0(N)^+$  is obtained by adding the Atkin-Lehner involutions to  $\Gamma_0(N)$ . In [8], it is shown that the automorphic scattering determinant  $\phi_N$  associated to  $\Gamma_0(N)^+$  is

$$\phi_N(s) = \frac{s}{s-1} \frac{\xi(2s-1)}{\xi(2s)} \cdot \frac{1}{N^s} \cdot \prod_{j=1}^r \frac{p_j^s + p_j}{p_j^s + 1},$$

where  $\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$  is the completed Riemann zeta function. We allow  $N = 1$  in which case  $\Gamma_0(N)^+ = \text{PSL}(2, \mathbb{Z})$ . Immediately, one can compute that  $\phi_N(1/2) = -1$ . For this example,  $\phi_N(s)$  possesses only one pole at  $s = 1$  and no zeros greater than  $1/2$ , and  $\text{sgd}(d(1)) = 1$ , thus verifying our main theorem.

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